Category Theory *

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These are some notes for the Cambridge Mathematical Tripos Part III course *Category Theory* in Michaelmas 2022. This is NOT a verbatim copy of the lectured material: I've edited the content to help me understand it. As a result, any errors are mine alone.

I'm actively maintaining these notes. If you want to report typos or mistakes, please email aik31@cam.ac.uk or message me on Discord at `FM22#2007`.

Contents

1 Basic definitions

Definition 1.1. A **category** $\mathcal{C}$ consists of

1. A collection $\operatorname{ob}\mathcal{C}$ of **objects** $A, B, C, \ldots$
2. A collection $\operatorname{mor}\mathcal{C}$ of **morphisms** $f, g, h, \ldots$
3. Operations dom, cod: $\operatorname{mor}\mathcal{C} \to \operatorname{ob}\mathcal{C}$, where $f : A \to B$ means that f is a morphism with $\operatorname{dom} f = A$ and $\operatorname{cod} f = B$.
4. An operation $A \to \mathbb{1}_A : \operatorname{ob}\mathcal{C} \to \operatorname{mor}\mathcal{C}$ where $\mathbb{1}_A : A \to A$.
5. A partial binary operation $(f, g) \to fg$ on $\operatorname{mor}\mathcal{C}$, where fg is defined iff $\operatorname{dom} f = \operatorname{cod} g$; then $\operatorname{dom} fg = \operatorname{dom} f$ and $\operatorname{cod} fg = \operatorname{cod} f$. This satisfies
   (a) $f\mathbb{1}_A = f = \mathbb{1}_A f$ (whenever defined).
   (b) $f(gh) = (fg)h$ (whenever defined).

*Based on the lectures under the same name taught by Prof. P. T. Johnstone in Michaelmas 2022.

Note that this is a set-free definition – we call $\mathcal{C}$ **small** if $\operatorname{ob}\mathcal{C}$ and $\operatorname{mor}\mathcal{C}$ are sets. Note also that we could define categories using only $\operatorname{ob}\mathcal{C}$ (and constant morphisms), and we could adopt the opposite composition convention. In this course, we do neither.

Examples 1.1.

1. **Set** is the category of sets and functions between them. Note morphisms $A \to B$ are *pairs* (f, B), where $f \subseteq A \times B$ is a (set-theoretic) function; the data about the codomain is technically required.

2. **Gp**, **Rng**, $\mathbf{Vect}_k$ are the categories of groups, rings and $k-$vector spaces, respectively, with the corresponding algebraic morphisms between them.

3. **Top**, **Met**, **Mfd** are the categories of topological spaces, metric spaces and smooth manifolds, respectively, with the corresponding regular maps between them.

4. **Htpy** is the category of topological spaces and homotopy classes of continuous maps between them.

5. Given an equivalence relation $\sim$ on $\operatorname{mor}\mathcal{C}$, we say $\sim$ is a **congruence** if $f \sim g$ implies that $\operatorname{dom} f = \operatorname{dom} g$, $hf \sim hg$ and $fk = gk$ (where defined). We can then form the **quotient category** $\mathcal{C}/\sim$ in the obvious way. **Htpy** is a quotient category of **Top**.

6. **Rel** is the category of sets and relations between them. $R : A \to B$ means $R \subseteq A \times B$, and $SR = \{(a, c) \mid \exists b \in B : (aRb \text{ and } bSc)\}$. Also **Part** is the category of sets and *partial* functions between them. We have **Set** inside **Part** inside **Rel**.

7. Given a category $\mathcal{C}$, define $\mathcal{C}^{\text{op}}$ to have the same objects and morphisms as $\mathcal{C}$, but with dom and cod interchanged, and fg in $\mathcal{C}^{\text{op}}$ meaning gf in $\mathcal{C}$.

8. A small category with one object $\bullet$ is a monoid (semigroup with identity but not necessarily inverses). In particular, groups are categories, where the group elements are morphisms $\bullet \to \bullet$.

9. A **groupoid** is a category in which all morphisms have inverses. An example is the **fundamental groupoid** $\pi_1(X)$ of a topological space X: the objects are the points of X and the morphisms $x \to y$ are homotopy classes (relative to endpoints) of paths $x \to y$. To see that this forms a groupoid, consider reversed paths.

10. If $\mathcal{C}$ has at most one morphism $A \to B$ for each pair of objects (A, B), then $\operatorname{mor}\mathcal{C}$ is a binary relation on $\operatorname{ob}\mathcal{C}$, where two objects are related if a morphism between them exists. This relation s reflexive (by identity) and transitive (by associativity); we call this kind of relation a **preorder**. In particular, posets are categories.

11. $\mathbf{Mat}_k$ (for a field k) has natural numbers as its objects, and morphisms $n \to p$ are $(p \times n)$ matrices with entries in k. Composition is matrix multiplication, which is indeed associative.

Definition 1.2. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of mappings $\operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$ and $\operatorname{mor}\mathcal{C} \to \operatorname{mor}\mathcal{D}$, both also denoted F, satisfying $\operatorname{dom}(Ff) = F\operatorname{dom}(f)$,

$\mathrm{cod}(Ff) = F\,\mathrm{cod}(f)$, $F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$ and $F(fg) = F(f)F(g)$ (whenever defined).

Examples 1.2.

1. *Forgetful functors*, which forget some of the structure: $\mathbf{Gp}\to\mathbf{Set}$, $\mathbf{Rng}\to\mathbf{Set}$, $\mathbf{Top}\to\mathbf{Set}$, $\mathbf{Rng}\to\mathbf{AbGrp}$, $\mathbf{Met}\to\mathbf{Top}, \ldots$

2. The free group functor $F : \mathbf{Set} \to \mathbf{Gp}$. Given a set A, there is a unique group FA containing a copy of A such that every (set) map $A \to G$, where G has a group structure, extends uniquely to a homomorphism $FA \to G$. We make this into a functor: given a set map $f : A \to B$, define $Ff : FA \to FB$ to be the unique homomorphism extending $A \xrightarrow{f} B \hookrightarrow FB$. For functoriality, note that, for $g : B \to C$, $F(gf)$ and $(Fg)(Ff)$ are both homomorphisms extending $A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow FC$.

3. The powerset functor $P : \mathbf{Set} \to \mathbf{Set}$ defined by $PA = \{\text{subsets of } A\}$ and, for $f : A \to B$, $Pf(A') = \{f(a) \mid a \in A'\}$.

4. The functor $P^* : \mathbf{Set} \to \mathbf{Set}^{op}$ defined by $P^*A = PA$ and, for $f : A \to B$, $P^*f(B') = \{a \in A \mid f(a) \in B'\}$.

   Call P^* a **contravariant functor Set** $\to$ **Set** (similarly call P a **covariant functor**).

5. The contravariant functor $(-)^* : \mathbf{Vect}_k^{op} \to \mathbf{Vect}_k$, where $V^* = \mathcal{L}(V, k)$ and, given $f : V \to W$, $f^* : W^* \to V^*$ sends forms $\theta : V \to k$ to their pullbacks along f, i.e., $f^*\theta = \theta \circ f$.

6. Writing **Cat** for the category of *small* categories and functors between them, have a functor $(-)^{op} : \mathbf{Cat} \to \mathbf{Cat}$. Note that this is covariant!

7. Viewing monoids and preorders as categories (as earlier), a functor between them is, respectively, a homomorphism and an order-preserving map.

8. Let G be a group, and view it as a category with object $\star$. A functor $G \to \mathbf{Set}$ is given by a set $A = F\star$ along with mappings $a \to ga : A \to A$ for each $g \in G$, each of which is a permutation. That is, a functor $G \to \mathbf{Set}$ is a permutation representation, or set action, of G. Similarly, a functor $G \to \mathbf{Vect}_k$ is a linear representation of G. Thus representation theory is *also* subsumed by category theory.

9. The fundamental groupoid construction is a functor $\pi_1(X) : \mathbf{Top} \to \mathbf{Cat}$ (in fact to the category of groupoids, defined in the obvious way). Similarly, have the fundamental group functor $\pi_1(X, x) : \mathbf{Top}_* \to \mathbf{Gp}$, where $\mathbf{Top}_*$ is the category of based topological spaces (X, x).

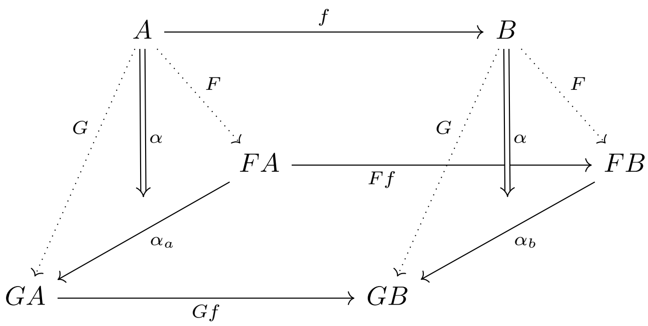
Definition 1.3. Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A **natural transformation** $\alpha : F \to G$ is a mapping $A \to \alpha_A : \mathrm{ob}\,\mathcal{C} \to \mathrm{mor}\,\mathcal{D}$ with $\alpha_A : FA \to GA$ for each object A of $\mathcal{C}$, such that the following 'naturality condition' holds: the square

$$\begin{array}{ccc} FA & \xrightarrow[Ff]{} & FB \\ {\scriptstyle \alpha_a}\big\downarrow & & {\scriptstyle \alpha_b}\big\downarrow \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

commutes for every $f : A \to B$ in $\operatorname{mor}\mathcal{C}$: that is, $\alpha_B(Ff) = (Gf)\alpha_A$.

Natural transformations can be composed: given another one $\beta : G \to H$, define $(\beta\alpha)_A = \beta_A \alpha_A$.

Here is a diagram summarising the definition:



Given categories $\mathcal{C}$ and $\mathcal{D}$, write $[\mathcal{C}, \mathcal{D}]$ for the category of functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

Examples 1.3.

1. For any vector space V there is a "natural" linear map $\alpha_V : V \to V^{**}$ given by $\alpha_v(x)(\theta) = \theta(x)$. In fact, $\alpha : V \to \alpha_V$ is a natural transformation $\mathbb{1}_{\mathbf{Vect}_k} \to (-)^{**}$, where $\mathbb{1}_{\mathbf{Vect}_k}$ is the identity functor.

2. Let $U : \mathbf{Gp} \to \mathbf{Set}$ be the forgetful functor on groups. We have a natural transformation $\mathbb{1}_{\mathbf{Set}} \to UF$ whose value at A is the inclusion $A \hookrightarrow FA$. The naturality condition holds was built into the definition of Ff.

3. There is a natural transformation $\mathbb{1}_{\mathbf{Set}} \to P$ whose value at A is the mapping $a \to \{a\}$.

4. If $f, g : P \rightrightarrows Q$ are maps of posets, a natural transformation $f \to g$ exists iff $f(p) \leq g(p)$ for all $p \in P$. The naturality condition is trivial in this instance as there is at most one morphism between any two objects.

5. If $u, v : G \rightrightarrows H$ are homomorphisms of groups, a natural transformation $u \to v$ is an element of H satisfying the naturality condition $hu(g) = v(g)h$ for all $g \in G$. Equivalently, h (or technically $\star \to h$) is a natural transformation $u \to v$ if and only if u and v are conjugate by h.

6. If A, B are permutation representations of a group G, viewed as functors $G \to \mathbf{Set}$, then a natural transformation $A \to B$ is a G-equivariant mapping $f : A \to B$; that is, one satisfying $f(ga) = gf(a)$ for $g \in G$.

7. The *Hurewicz homomorphism* $h_n : \pi_n(X, x) \to H_n(X)$ gives a natural transformation $\pi_n \to H_n U$, where U is the forgetful functor $\mathbf{Top}_* \to \mathbf{Top}$.

Definition 1.4. An **isomorphism** of categories is a functor with a two-sided inverse.

Example 1.1. $\mathbf{Rel} \cong \mathbf{Rel}^{\text{op}}$ via the functor taking $A \to A$ and $R \to R^\circ = \{(a,b) \mid (b,a) \in R\}$.

We want to define a weaker notion of equivalence. To do this we want to define natural isomorphisms.

Definition 1.5. A **natural isomorphism** is a natural transformation which is an isomorphism (in $[\mathcal{C}, \mathcal{D}]$).

Lemma 1.1. *Let $\alpha : F \to G$ be a natural transformation between functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$. Then α is a natural isomorphism iff α_A is an isomorphism (in $\mathcal{D}$) for each $A \in \mathcal{C}$.*

Proof.
$\Rightarrow$: Obvious.

$\Leftarrow$: Let each α_A have inverse β_A. We need to show that the β_A form a natural transformation. It suffices to check naturality; given $f : A \to B$ in $\mathcal{C}$, have

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle\alpha_A}\downarrow\uparrow{\scriptstyle\beta_A} & & {\scriptstyle\alpha_B}\downarrow\uparrow{\scriptstyle\beta_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

so $(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$. □

Definition 1.6. An **equivalence** between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$. Write $\mathcal{C} \simeq \mathcal{D}$.

A **categorical property** of categories is a property P which is preserved by equivalence.

Equivalence of categories is clearly an equivalence relation, as we can compose natural transformations.

Examples 1.4.

1. Let $\mathbf{Set}_x$ be the category of pointed sets. Then $\mathbf{Set}_x \simeq \mathbf{Part}$. Indeed, define $F : \mathbf{Set}_x \to \mathbf{Part}$ by $F(A,a) = A \setminus a$ and, for $f : (A,a) \to (B,b)$,

$$Ff(x) = \left\{ \begin{matrix} f(x) & x \neq a \\ \text{undefined} & x = a \end{matrix} \right\}.$$

Also define $G : \mathbf{Part} \to \mathbf{Set}_x$ by $G(A) = (A \cup \{A\}, A)$ and, for $f : A \rightharpoonup B$,

$$Gf(x) = \left\{ \begin{matrix} f(x) & x \in A \text{ and } f(x) \text{ defined} \\ B & \text{otherwise} \end{matrix} \right\}.$$

FG is the identity functor on **Part**; GF is not the identity, but there is an obvious natural isomorphism $(A,a) \to GF(A,a) = ((A \setminus a) \cup \{A\}, A)$.

2. Let $\mathbf{fdVect}_k$ be the category of finite-dimensional vector spaces over k. Then $\mathbf{fdVect} \simeq \mathbf{fdVect}^{\mathrm{op}}$. Indeed, take both F and G to be $(-)^*$, and α and β to both be the double dual natural transformation (seen earlier).

3. $\mathbf{fdVect}_k \simeq \mathbf{Mat}_k$ (defined earlier). Indeed, have $F : \mathbf{Mat}_k \to \mathbf{fdVect}_k$ with $Fn = k^n$ and FA the map represented by A wrt the standard bases. To define G, first fix a basis for each object of $\mathbf{fdVect}_k$, choosing the standard bases for k^n. Then define $GV = \dim V$, and let $G(V \xrightarrow{f} W)$ be the matrix of f wrt the chosen bases. Then GF is the identity; FG is not, but the chosen bases define isomorphisms $k^{\dim V} \to V$ which form a natural transformation $FG \to 1_{\mathbf{fdVect}_k}$.

Definition 1.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

F is **faithful** if, for all morphisms f, g in $\mathcal{C}$ with $\operatorname{dom} f = \operatorname{dom} g$, $\operatorname{cod} f = \operatorname{cod} g$, and $Ff = Fg$, in fact $f = g$.

F is **full** if, given $g : FA \to FB$ in $\mathcal{D}$, there is some $f : A \to B$ in $\mathcal{C}$ with $Ff = g$.

F is **essentially surjective** if, for every object B of $\mathcal{D}$, there is some object A of $\mathcal{C}$ with $FA \cong B$; that is, if it is surjective on isomorphism classes of objects.

Note that, if F is full and faithful, it's automatically injective on isomorphism classes of objects: given an isomorphism $g : FA \xrightarrow{\cong} FB$ in $\mathcal{D}$, the unique $f : A \to B$ with $Ff = g$ is an isomorphism as it has inverse the unique $h : B \to A$ with $Fh = g^{-1}$. This is because the unique preimages of 1_{FA} and 1_{FB} must respectively be 1_A and 1_B.

Definition 1.8. A subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is **full** if the inclusion functor $\mathcal{C}' \to \mathcal{C}$ is full.

That is, $\mathcal{C}'$ is full if every morphism in $\mathcal{C}$ between objects of $\mathcal{C}'$ is also a morphism of $\mathcal{C}'$.

Example 1.2. **Gp** is a full subcategory of **Mon** (the category of monoids), but **Mon** is not a full subcategory of **SGp** (the category of semigroups) as semigroup morphisms need not preserve 1.

Lemma 1.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ iff F is full, faithful and essentially surjective.*

Proof.
$\Rightarrow$: Let G, α, β as in the definition of equivalence. F is essentially surjective via $\alpha\beta$. For any $f : A \to B$ in $\mathcal{C}$, $f = \alpha_B^{-1}(GFf)\alpha_A$ by naturality, so F is faithful. Given $g : FA \to FB$ in $\mathcal{D}$, define $f = \alpha_B^{-1}(Gg)\alpha_A$; then $GFf = Gg$, but G is faithful (as we just showed) so $Ff = g$. Hence F is full.

$\Leftarrow$: For $B \in \operatorname{ob}\mathcal{D}$, choose $GB \in \operatorname{ob}\mathcal{C}$ and an isomorphism $\beta_B : FGB \to B$ in $\mathcal{D}$ (as F is essentially surjective). For $g : B \to C$ in $\mathcal{D}$, define $Gg : GB \to GC$ to be the unique morphism whose image under F is $\beta_C^{-1} g \beta_B : FGB \to FGC$. Functoriality follows from faithfulness: given $h : C \to D$, $(Gg)(Gh)$ and $G(gh)$ have the same image under F, so they are equal. By construction, β is a natural isomorphism $FG \to 1_{\mathcal{D}}$. Then take $\alpha_A : A \to GFA$ to be the unique morphism whose image under F is β_{FA}^{-1}; α_A is an isomorphism since F is full and faithful.

Naturality squares for α are mapped by F to naturality squares for β^{-1}, so they commute. Hence α is a natural isomorphism. $\square$

Definition 1.9. A category $\mathcal{C}$ is **skeletal** if $A \cong B$ implies $A = B$. A **skeleton** of $\mathcal{C}$ is a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ containing exactly one object from each isomorphism class of $\mathrm{ob}\,\mathcal{C}$.

Example 1.3. $\mathbf{Mat}_k$ is skeletal; $\{k^n \mid n \in \mathbb{N}\}$ is a skeleton for $\mathbf{fdVect}_k$.

Definition 1.10. Let $f : A \to B$ be a morphism in a category. Say f is a **monomorphism** $f : A \hookrightarrow B$ (is **monic**) if $fg = fh \implies g = h$ (when defined). Say f is an **epimorphism** $f : A \twoheadrightarrow B$ (is **epic**) if it a monomorphism in $\mathcal{C}^{\mathrm{op}}$, that is, if $gf = hf \implies g = h$ (when defined).

$\mathcal{C}$ is **balanced** if all monic, epic morphisms in $\mathcal{C}$ are in fact isomorphisms.

Examples 1.5.

1. In **Set**, injective implies monic, and surjective implies epic (trivially). Monic implies injective: consider morphisms $\{\bullet\} = 1 \to A$; also, epic implies surjective: consider morphisms $A \to 2 = \{0, 1\}$. Hence **Set** is balanced.

2. In **Gp**, again have monic iff injective (use $\mathbb{Z}$ in place of 1) and epic iff surjective (use free product with amalgamation). Hence **Gp** is also balanced.

3. In **Rng**, again have monic iff injective (use maps to $\mathbb{Z}[X]$). However, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism. Thus **Rng** is not balanced.

4. In **Top**, have monic iff injective and epic iff surjective, but not all bijective maps are isomorphisms; in fact, **Top** is not balanced.

5. In a preorder, every morphism is monic and epic (by uniqueness), so a preorder is balanced iff it is an equivalence relation.

2 The Yoneda Lemma

Definition 2.1. A category $\mathcal{C}$ is **locally small** if, for every $A, B \in \mathrm{ob}\,\mathbb{C}$, the morphisms $A \to B$ form a set, denoted $\mathcal{C}(A, B)$.

Example 2.1. Most categories we have seen so far,(e.g., where the objects are sets and morphisms structure preserving maps between them) are all locally small. The only exception is $[\mathcal{C}, \mathcal{D}]$: this is locally small if $\mathcal{C}$ and $\mathcal{D}$ are locally small, but it's not locally small in general.

For the remainder of this chapter, assume $\mathcal{C}$ is locally small.

If $A \in \mathrm{ob}\mathcal{C}$, there is a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ taking $B \to \mathcal{C}(A, B)$ and $g : B \to C$ to $(f \to gf) : \mathcal{C}(A, B) \to \mathcal{C}(A, C)$. Functoriality of $\mathcal{C}(A, -)$ follows from associativity of composition. Similarly, the map $A \to \mathcal{C}(A, B)$ defines a functor $\mathcal{C}(-, B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Theorem 2.1 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a locally small category. Fix $A \in \mathrm{ob}\,\mathcal{C}$ and a functor $F : \mathcal{C} \to$ **Set**. Then $[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F)$ (that is, the set of natural transformations $\mathcal{C}(A, -) \to F$) is in bijection with elements of FA.*

Proof. Given $\alpha : \mathcal{C}(A, -) \to F$, define $\Phi(\alpha) = \alpha_A(\mathbb{1}_A) \in FA$. Given $x \in FA$, define $\Psi(x) : \mathcal{C}(A, -) \to F$ by $\Psi(x)_B(f : A \to B) = (Ff)(x) \in FB$. Then $\Psi(x)$ is natural since F is a functor.

Now, $\Phi\Psi(x) = \Psi(x)_A(\mathbb{1}_A) = F(\mathbb{1}_A)(x) = x$ for $x \in FA$. Then fix $\alpha : \mathcal{C}(A, -) \to F$; have

$$\Psi\Phi(\alpha)_B(f : A \to B) = Ff(\Phi(x)) = Ff\alpha_A(\mathbb{1}_A) = \alpha_B\mathcal{C}(A, f)(\mathbb{1}_A) = \alpha_B(f).$$

Thus $\Psi\Phi(\alpha) = \alpha$. □

The Yoneda lemma allows us to identify natural transformations from $\mathcal{C}(A, -)$ with their values at $\mathbb{1}_A$; the reverse identification is done using the map Ψ in the proof.

Example 2.2. Let $\mathcal{C} = G$ be a group. A natural transformation from $G(\bullet, -)$ is a G-equivariant map φ from the regular representation, and such a map is determined by $\varphi(1)$: indeed, $\varphi(g) = g \cdot \varphi(1)$.

Corollary 2.2. *The mapping $A \to \mathcal{C}(A, -)$ yields a full and faithful functor $\mathcal{C}^{op} \to [\mathcal{C}, \textbf{Set}]$.*

Proof. Applying the Yoneda lemma with $F = \mathcal{C}(B, -)$ gives a bijection of $\mathcal{C}(B, A)$ with the set of natural transformations $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$. This defines a functor by associativity of composition in $\mathcal{C}$. □

Using $A \to \mathcal{C}(-, A)$ instead, we also get a full and faithful functor $\mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$. These are both called the **Yoneda embedding**, and allow us to regard $\mathcal{C}$, up to equivalence, as a full subcategory of $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$.

The Cayley representation theorem for groups (every group is isomorphic to a subgroup of a permutation group) is an instance of this embedding, as is the Dedekind representation theorem for posets (every poset is isomorphic to a sub-poset of a powerset).

The Yoneda embedding allows us to make sense of what it means for Φ and Ψ to be *natural*. Indeed, suppose $\mathcal{C}$ is small; then $[\mathcal{C}, \mathbf{Set}]$ is locally small. This allows us to define functors $\mu, \nu : \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightrightarrows \mathbf{Set}$, where $\mu(A, F) = FA$ and $\nu(A, F) = [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F)$; these map morphisms in an 'obvious' way. The claim is then that Φ and Ψ yield a natural isomorphism of μ and ν.

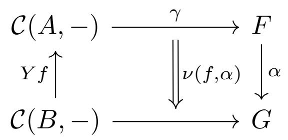
Indeed, let μ map $(f : A \to B, \alpha : F \to G)$ to the diagonal of the naturality square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \alpha_A}\downarrow & & {\scriptstyle \alpha_B}\downarrow \\ GA & \xrightarrow{Gf} & GB \end{array}$$

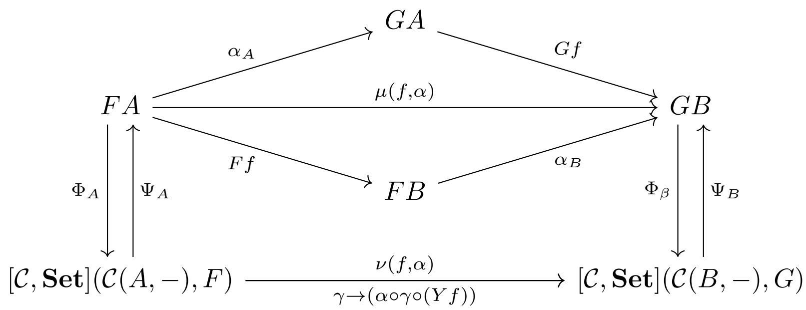
Then define ν to be the composite

$$\nu : \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}, \mathbf{Set}]^{\text{op}} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-,-)} \mathbf{Set}$$

where Y is the Yoneda embedding $A \to \mathcal{C}(A,-)$; in this case, Yf maps $g : B \to C$ to $gf : A \to C$. Then ν maps (f, α) to the composition action below:



The naturality claim is then that the following diagram commutes:



Thus we have

Corollary 2.3. *The bijection defined by Φ and Ψ in the Yoneda lemma is natural in F and A.*

Proof. Translating the diagram above into elementary terms, we get a naturality claim that makes sense even when $\mathcal{C}$ is only locally small:

Naturality claim: Given $f : A \to B$ in $\mathcal{C}$, $\alpha : F \to F'$ in $[\mathcal{C}, \mathbf{Set}]$, and $x \in FA$, then

$$((Gf) \circ \alpha_A)(x) = (\alpha_B \circ (Ff))(x) = \Phi(\alpha \circ \Psi(x) \circ (Yf)).$$

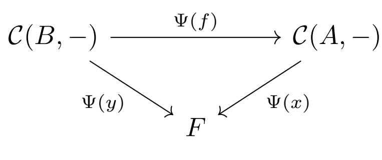
This equation is easy to verify. □

Definition 2.2. A functor $\mathcal{C} \to \mathbf{Set}$ is **representable** if it is isomorphic to $\mathcal{C}(A,-)$ for some $A \in \operatorname{ob}\mathcal{C}$. A **representation** of F is then a pair (A, x) where $A \in \operatorname{ob}\mathcal{C}$, and $x \in FA$ yields a natural *isomorphism* $\Psi(x) : \mathcal{C}(A,-) \to F$. Then x is called a **universal element** of F.

Note that $\Psi(x)$ is a natural isomorphism is iff there is *precisely* one morphism $f : A \to B$ for each possible value of $Ff(a)$.

Corollary 2.4. *Let (A, x) and (B, y) be two representations of $F : \mathcal{C} \to$ **Set**. Then there is a unique isomorphism $f : A \to B$ such that $Ff(x) = y$.*

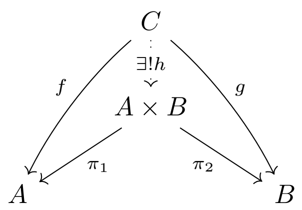
Proof. The equation $Ff(x) = y$ is equivalent to saying that the diagram



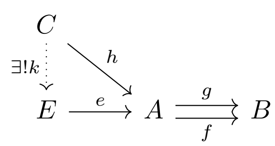
commutes. Therefore f must be the unique morphism $A \to B$ whose image $\Psi(f)$ under the Yoneda embedding is $\Psi(x)^{-1}\Psi(y)$. This is indeed an isomorphism. □

Examples 2.1.

1. The forgetful functor $\mathbf{Gp} \to \mathbf{Set}$ is representable by $(\mathbb{Z}, 1)$.
2. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is representable by $(\{\star\}, \star)$.
3. The functor $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is representable by $(2 = \{0,1\}, \{1\})$. However, $P : \mathbf{Set} \to \mathbf{Set}$ is *not* representable: for any $A \in \mathbf{Set}$, we have that $\mathbf{Set}(A, 1)$ is a singleton, but $P(1)$ has two elements. More explicitly, fix a universal element $S \subset A$. If $S = \emptyset$, there is no map $f : A \to 1$ with $f(S) = 1$; if $S \neq \emptyset$, there is no map $f : A \to 1$ with $f(S) = \emptyset$.
4. The composite of the dual-space functor $(-)^* : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k$ with $U : \mathbf{Vect}_k \to \mathbf{Set}$ is representable by $(k, 1_k)$.
5. A group G has a unique (up to isomorphism) representable functor $(\bullet, g)$, which is the left regular (Cayley) representation (with the set map twisted by g).
6. Let A and B be two objects of $\mathcal{C}$. We have a functor $\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ mapping $C \to \mathcal{C}(C, A) \times \mathcal{C}(C, B)$ (and mapping morphisms using the obvious pair of compositions). Suppose this functor is representable. Then the representation consists of a **product**: an object $A \times B$ equipped with morphisms $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ such that, given maps $f : C \to A$ and $g : C \to B$, there is a unique $h : C \to A \times B$ such that the following diagram commutes:

   Dually, we have the notion of **coproduct**, or sum, $A \xrightarrow{\nu_1} A + B \xleftarrow{\nu_2} B$, representing $\mathcal{C}(A, -) \times \mathcal{C}(B, -)$. These exist in **Set**, **Gp**, **Rng**, **Top**, ... In a poset, products and coproducts give suprema and infima, respectively.
7. Let $f, g : A \rightrightarrows B$ be a parallel pair of morphisms in $\mathcal{C}$. A representation of the functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ sending $C \to \{h : C \to A \mid fh = gh\}$ is an **equaliser**: an object E along with a map $e : E \to A$ such that $fe = ge$, and, for all $h : C \to A$ with $fh = gh$, there is a unique $k : C \to E$ such that the following diagram commutes:

Note that e must be monic. A **regular monomorphism** is one which occurs as an equaliser. Dually (representing $\{h : A \to C \mid hf = hg\}$), we get **coequalisers** and **regular epimorphisms**.

Note that a regular monomorphism which is an epimorphism must be an isomorphism, since an epic equaliser must equalise some pair (f, f).

The notions of product and equaliser (and their duals) are defined independently of the functors they represent. In particular, these notions are defined for all categories.

Definition 2.3. Let $\mathcal{G}$ be a class of objects of $\mathcal{C}$.

(a) $\mathcal{G}$ is a **separating family** for $\mathcal{C}$ if the functors $\mathcal{C}(G, -)$ for $G \in \mathcal{G}$ are **collectively faithful**: that is, if $f, g : A \rightrightarrows B$ in $\mathcal{C}$ satisfies $fh = gh$ for all $h : G \to A$ with $G \in \mathcal{G}$, then $f = g$.

(b) $\mathcal{G}$ is a **detecting family** if the $\mathcal{C}(\mathcal{G}, -)$ **collectively reflect isomorphisms**: that is, if $f : A \to B$ is such that every $h : G \to B$ with $G \in \mathcal{G}$ factors uniquely through f, then f is an isomorphism.

If $\mathcal{G} = \{G\}$, call G a **separator** or **detector**, respectively.

Lemma 2.5.

(i) If $\mathcal{C}$ has equalisers (for every parallel pair of morphisms), then every detecting family is separating.

(ii) If $\mathcal{C}$ is balanced, then every separating family is detecting.

Proof.

(i) Suppose $\mathcal{G}$ is detecting, and let $f, g : A \rightrightarrows B$ satisfy $fh = gh$ for $h : G \to A$ with $G \in \mathcal{G}$. Let $e : E \to A$ equalise (f, g); then every $g : h \to A$ factors uniquely through e, so e is an isomorphism. Hence $f = g$.

(ii) Suppose $\mathcal{G}$ is separating, and fix $f : A \to B$ such that every $k : G \to C$ with $G \in G$ factors uniquely through f. If $g, h : C \rightrightarrows A$ satisfy $fg = fh$, then $fgk = fhk$, so (by unique factorisation) $gk = hk$. Hence $g = h$.

Similarly, if $l, m : B \rightrightarrows D$ satisfy $lf = mf$, then any $n : G \to B$ with $G \in \mathcal{G}$ satisfies $ln = mn$, so $l = m$. Hence f is also epic, so it is an isomorphism.

□

Examples 2.2.

1. $\mathbb{Z}$ is a separator and detector for **Gp**, and $\mathbb{Z}[X]$ is a separator and detector for **Rng**. In general, the free object on one generator gives a separator and detector for any "algebraic" category.

2. Let $\mathcal{C}$ be small. Then $\{\mathcal{C}(A, -) \mid A \in \operatorname{ob}\mathcal{C}\}$ forms a separating and detecting *set* in $[\mathcal{C}, \mathbf{Set}]$ (note the objects don't form a set).

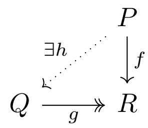
3. In **Top**, $\{\bullet\}$ is a separator, but **Top** has no detecting *set* of objects. Indeed, any such set has a (strict) upper bound κ on the cardinalities of its elements. Then there are spaces X_κ and Y_κ such that the identity map

Id : $X_\kappa \to Y_\kappa$ is continuous, but not a homeomorphism, yet Id becomes a homeomorphism when restricted to any subset of X_κ with strictly smaller cardinality.

4. Let $\mathcal{C}$ be the category of pointed connected CW complexes, with homotopy classes of continuous maps between them. Whitehead showed that $\{S^n \mid n \geq 0\}$ is a detecting set for this category. Freyd showed there is no faithful functor $\mathcal{C} \to \mathbf{Set}$, which implies that there is no separating set for $\mathcal{C}$.

A functor of the form $\mathcal{C}(A, -)$ preserves monomorphisms, by definition.

Definition 2.4. $P \in \operatorname{ob}\mathcal{C}$ is **projective** if $\mathcal{C}(P, -)$ **preserves epimorphisms**. That is, given a diagram as below, there is some $h : P \to Q$ with $gh = f$.



Dually, P is **injective** in $\mathcal{C}$ if it is projective in $\mathcal{C}^{\mathrm{op}}$, that is, if $\mathcal{C}(-, P)$ preserves epimorphisms.

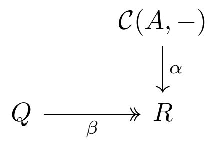
If P preserves some class $\mathcal{E}$ of epimorphisms, call P **$\mathcal{E}$-projective**.

In $[\mathcal{C}, \mathbf{Set}]$, say $\alpha : F \to G$ is a **pointwise** epimorphism if α_A is surjective for all A (note α is automatically epic).

In fact, we will show that, in this category, all epimorphisms are in fact pointwise epimorphisms. This will allow us to delete "pointwise" from the following two results.

Corollary 2.6. *For any small $\mathcal{C}$, representable functors are pointwise projective in $[\mathcal{C}, \mathbf{Set}]$.*

Proof. It suffices to show the functors $\mathcal{C}(A, -)$ are projective. Given a diagram



let $\alpha = \Psi(x)$ for some $x \in RA$. Since β_A is surjective, there is some $y \in QA$ satisfying $\beta_A(y) = x$. Then $\Psi(y) : \mathcal{C}(A, -) \to Q$ is the required factoring map. □

Proposition 2.7. *For any small $\mathcal{C}$, the category $[\mathcal{C}, \mathbf{Set}]$ has **enough pointwise projectives**: that is, for any $F : \mathcal{C} \to \mathbf{Set}$, there is a pointwise epimorphism $\alpha : P \twoheadrightarrow F$ with P pointwise projective.*

Proof. Set

$$P \coloneqq \coprod_{\{(A,x)\}} \mathcal{C}(A, -),$$

where A runs over $\operatorname{ob}\mathcal{C}$ and x runs over FA. This is an infinitary version of the coproduct in **Set**. Explicitly,

$$PB = \coprod_{\{(A,x)\}} \mathcal{C}(A, B),$$

and so morphisms $P \to G$ correspond to families of morphisms

$$\gamma^{(A,x)} : \mathcal{C}(A, -) \to G.$$

Clearly, each component of P is representable, and so pointwise projective. Since a coproduct of projective objects is projective, P itself is pointwise projective. Then (the disjoint union of) the family $\gamma^{(A,x)} := \Psi_A(x) : \mathcal{C}(A, -) \to F$ is pointwise epic, since

$$x = \Psi_A(x)(\mathbb{1}_A) = \gamma_A^{(A,x)}(\mathbb{1}_A).$$

□

3 Adjunctions

Definition 3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An **adjunction** between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, together with, for each $A \in \operatorname{ob}\mathcal{C}$ and $B \in \operatorname{ob}\mathcal{D}$, a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and $A \to GB$ in $\mathcal{C}$ which is natural in A and B.

If $\mathcal{C}$ and $\mathcal{D}$ are locally small, the naturality condition is that $\mathcal{D}(F-, -)$ and $\mathcal{C}(-, G-)$ are naturally isomorphic functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to$ **Set**. Spelling it out, we want bijections $\alpha_{A,B}$ such that, for morphisms $f : A \to C$ in $\mathcal{C}$ and $g : B \to D$ in $\mathcal{D}$, the diagram below commutes:

$$\begin{array}{ccc}
\mathcal{C}(FA, B) & \xrightarrow{h \to gh(Ff)} & \mathcal{C}(FC, D) \\
{\scriptstyle \alpha_{A,B}} \big\downarrow {\scriptstyle \simeq} & & {\scriptstyle \alpha_{C,D}} \big\downarrow {\scriptstyle \simeq} \\
\mathcal{D}(A, GB) & \xrightarrow[h \to (Gg)hf]{} & \mathcal{D}(C, GD)
\end{array}$$

It is easier to check naturality in the first and second indices separately; this suffices since we can decompose morphisms in $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ as

$$(f, g) = (f, 1)(1, g).$$

We say F is **left adjoint** to G and G is **right adjoint** to F, and we write $F \dashv G$.

Examples 3.1.

1. The free group functor F : **Set** $\to$ **Gp** is left adjoint to U : **Gp** $\to$ **Set**. Indeed, by definition, homomorphisms $FA \to G$ correspond to functions $A \to UG$. Naturality in A follows from the definition of F as a functor, and naturality in G is trivial.

   This generalises to other free functors (on rings, modules, etc.)

2. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has a left adjoint D. Indeed, set $DA = A$ to be A equipped with the discrete topology. Then any function $A \to UX$ becomes continuous as a function $DA \to X$; since this bijection doesn't change the underlying mapping, it is clearly natural.

   U also has a right adjoint I, where IA is A equipped with the indiscrete topology. This time, set maps *into* IA are continuous, so adjointness follows by the same argument as for D.

3. The functor $\mathrm{ob} : \mathbf{Cat} \to \mathbf{Set}$ has a left adjoint D, where DA is the discrete category with $\mathrm{ob}\, DA = A$. It also has a right adjoint I where $\mathrm{ob}\, IA = A$ and $\mathrm{mor}\, IA = A \times A$ (that is, there is exactly one morphism from any object to any other). These are similar to the functors $D, I : \mathbf{Set} \to \mathbf{Top}$.

   D has also a left adjoint π_0, where $\pi_0\mathcal{C}$ is the set of **connected components** of $\mathcal{C}$: that is, the quotient of $\mathrm{ob}\, \mathcal{C}$ by the smallest equivalence relation identifying $\mathrm{dom}\, f$ and $\mathrm{cod}\, f$ for $f \in \mathrm{mor}\, \mathcal{C}$. Indeed, any functor $\mathcal{C} \to DA$ is constant on any class in $\pi_0\mathcal{C}$.

   We thus have a chain of adjoints $\pi_0 \dashv D \dashv \mathrm{ob} \dashv I$.

4. Fix a set A. The operation $(-) \times A$ is a functor $\mathbf{Set} \to \mathbf{Set}$. This functor has a right adjoint $\mathbf{Set}(A, -)$ acting on morphisms by currying: given $f : C \times A \to B$, define $\lambda f : C \to \mathbf{Set}(A, B)$ by $\lambda f(c)(a) = f(c, a)$.

   More generally, let $\mathcal{C}$ be a category with (binary) products. Then $\mathcal{C}$ is **Cartesian closed** if, for any $A \in \mathrm{ob}\, \mathcal{C}$, the functor $(-) \times A : \mathcal{C} \to \mathcal{C}$ has a right adjoint $(-)^A$.

   For example, $\mathbf{Cat}$ is Cartesian closed: set $\mathcal{D}^{\mathcal{C}} := [\mathcal{C}, \mathcal{D}]$, and curry morphisms in the obvious way.

5. Let $M = \{1, i\}$ be a 2-element monoid with an idempotent $i^2 = i$. A functor $M \to \mathbf{Set}$ is a set A with an idempotent map $i : A \to A$. Picking the identity map, we get a functor $F : \mathbf{Set} \to [M, \mathbf{Set}]$ mapping A to $(A, \mathbb{1}_A)$; we also have a functor $G : [M, \mathbf{Set}] \to \mathbf{Set}$ mapping (A, i) to the set of fixed points $\{a \in A \mid ia = a\}$.

   These are adjoint both ways around: $F \dashv G$ since a morphism $FA \to (B, e)$ takes values in $G(B, e)$, and $G \dashv F$ since any $f : (A, e) \to FB$ satisfies $f(a) = f(ia)$, so f is determined by $f_{G(A,e)}$. However, F and G are clearly not equivalent.

6. Consider the category $D1$ (the discrete category on one element); observe that, for any other category $\mathcal{C}$, there is a unique functor $\mathcal{C} \to D1$. A right adjoint for the unique functor $\mathcal{C} \to \mathcal{D}1$ picks out a **terminal object** of $\mathcal{C}$: an object T such that there is a unique morphism to it from any $S \in \mathrm{ob}\, \mathcal{C}$; equivalently, $\mathcal{C}(S, T)$ is a singleton. A left adjoint for $\mathcal{C} \to D1$ then picks out an **initial object** of $\mathcal{C}$ (unique morphism *from* it to any object).

7. Fix a map $f : A \to B$ of sets. Then the poset functor $Pf : PA \to PB$ is left adjoint to $P^*f : PB \to PA$, since $f(A') \subseteq B'$ iff $A' \subseteq f^{-1}(B')$, giving a bijection. Note that the naturality condition here is trivial.

8. Fix $R \subseteq A \times B$. Define mappings $(-)^r : PA \to PB$ and $(-)^l : PB \to PA$ by

$$S^r = \{b \in B \mid \forall a \in S : aRb\} \text{ and } T^l = \{a \in A \mid \forall b \in T : aRb\}.$$

These are contravariant functors, and

$$T \subseteq S^r \iff S \times T \subseteq R \iff S \subseteq T^l.$$

We say $(-)^l$ and $(-)^r$ are adjoint to each other **on the right**; this type of adjunction is called a **Galois connection**.

9. $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is self-adjoint on the right, since functions $A \to PB$ correspond to relations between A and B, which correspond to functions $B \to PA$, and this correspondence is in fact natural.

10. Similarly, $(-)^* : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k$ is self-adjoint on the right, since linear maps $V \to W^*$ and $W \to V^*$ correspond to bilinear forms on $V \times W$.

Theorem 3.1. *Fix a functor $G : \mathcal{D} \to \mathcal{C}$. For each $A \in \operatorname{ob}\mathcal{C}$, let $(A \downarrow G)$ be the category whose objects are pairs (B, f) with $B \in \operatorname{ob}\mathcal{D}$ and $f : A \to GB$ in $\mathcal{C}$, and whose morphisms $(B, f) \to (B', f')$ are morphisms $g : B \to B'$ in $\mathcal{D}$ such that*

$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ & \searrow^{f'} & \downarrow{\scriptstyle Gg} \\ & & GB' \end{array}$$

commutes.

Then a left adjoint for G corresponds to a choice of initial object of $(A \downarrow G)$ for each $A \in \operatorname{ob}\mathcal{C}$.

The categories $(A \downarrow G)$ are called **comma categories** or **down arrow categories**.

Proof. Suppose $F \dashv G$. For $A \in \operatorname{ob}\mathcal{C}$, let $\eta_A : A \to GFA$ be the morphism corresponding to $\mathbb{1}_A$. Then (FA, η_A) is initial in $(A \downarrow G)$: indeed, if $g : FA \to B$, then $(Gg)\eta_A$ corresponds to $g\mathbb{1}_{FA} = g$, so there is a unique g such that

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & \searrow^{f} & \downarrow{\scriptstyle Gg} \\ & & GB \end{array}$$

commutes.

Conversely, suppose we are given an initial object (FA, η_A) in each $(A \downarrow G)$. To define F on a morphism $f : A \to A'$, let $Ff : FA \to FA'$ be the unique one making the diagram below commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ FA & \xrightarrow[\eta_{A'}]{} & GFA' \end{array}$$

Functoriality follows from uniqueness: given $f' : A' \to A''$, $F(f'f)$ and $F(f')F(f)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''} f'f)$ in $(A \downarrow G)$, so they must be equal.

Given $f : A \to GB$, its transpose is the $g : FA \to B$ with $(Gg)\eta_A = f$. This is natural in A since η is a natural transformation $1\!\!1_{\mathcal{C}} \to GF$, and natural in B by definition. □

Corollary 3.2. *If $G : \mathcal{D} \to \mathcal{C}$ has left adjoints F and F', then there is a canonical natural isomorphism $\alpha : F \to F'$.*

Proof. (FA, q_A) and $(F'A, q'A)$ are both initial in $(A \downarrow G)$, so there is a unique isomorphism $\alpha_A : (FA, q_A) \to (F'A, q'_A)$: indeed, the composite with the unique map the other way must be the identity, by uniqueness.

The map $A \to \alpha_A$ is natural since, given any $f : A \to A'$, the square

$$\begin{array}{ccc} (FA, q_A) & \xrightarrow{Ff} & (FA', q_{A'}) \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_{A'}} \\ (F'A, q'_A) & \xrightarrow[F'f]{} & (F'A', q'_{A'}) \end{array}$$

commutes. □

Lemma 3.3. *Suppose we have adjunctions $F \dashv G$ and $H \dashv K$ as below:*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathcal{E}$$

Then $HF \dashv GK$.

Proof. Given $A \in \operatorname{ob} \mathcal{C}$ and $C \in \operatorname{ob} \mathcal{E}$, we have bijections between morphisms $HFA \to C$, morphisms $FA \to KC$, and morphisms $A \to GKC$. These are natural in A and C. □

Corollary 3.4. *Suppose we have the commuting square of categories and functors*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle H} \\ \mathcal{E} & \xrightarrow[K]{} & \mathcal{F} \end{array}$$

where each functor has a left adjoint. Then the square of left adjoints commutes up to natural isomorphism.

Proof. Both compositions around the square of left adjoints are left adjoint to $HF = KG$, so they are naturally isomorphic. □

The natural transformation $\eta : 1\!\!1_{\mathcal{C}} \to GF$ corresponding to $1\!\!1_{FA}$ (appearing in the proof of the last theorem) is called the **unit** of $F \dashv G$. Dually, we have the **counit** $\varepsilon : FG \to 1\!\!1_{\mathcal{D}}$ of the adjunction: $\varepsilon_B : FGB \to B$ corresponds to $1\!\!1_{GB}$.

Theorem 3.5. *Fix $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Then specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta : \mathbb{1}_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to \mathbb{1}_{\mathcal{D}}$ such that the* ***triangle identites*** *hold: that is, the diagrams below commute.*

$$\begin{array}{ccc} F & \xrightarrow{Fg} & FGF \\ & \searrow^{\mathbb{1}_F} & \downarrow{\scriptstyle \varepsilon_F} \\ & & F \end{array} \qquad\qquad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & \searrow^{\mathbb{1}_G} & \downarrow{\scriptstyle G\varepsilon} \\ & & G \end{array}$$

Here, ε_F (for example) refers to the natural transformation whose value at A is ε_{FA}.

Proof. Given an adjunction $F \dashv G$, define η and ε to be the unit and counit of the adjunction, respectively. By naturality of the morphism bijection, $\varepsilon_{FA}(Fg_A) : FA \to GFA \to FA$ corresponds to $\mathbb{1}_{GFA}\eta_A : A \to GFA \to GFA$. The other identity is dual.

Conversely, suppose we have η and ε satisfying the triangular identities. Given $g : FA \to B$, define $\Phi(g) = (Gg)\eta_A : A \to GFA \to GB$; given $f : A \to GB$, define $\Psi(f) = \varepsilon_B(Ff) : FA \to FGB \to GB$. These are natural since η and ε are; it remains to show they are inverses.

Indeed, by the triangular identities,

$$\Phi\Psi(f) = \eta_A G\Psi(f) = \eta_A(GFf)G\varepsilon_B = f\eta_{GB}G\varepsilon_B = f.$$

The other identity is dual. $\square$

Are equivalences of categories adjoint?

Lemma 3.6. *Let $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}, \alpha : \mathbb{1}_{\mathcal{C}} \xrightarrow{\simeq} GF, \beta : FG \xrightarrow{\simeq} \mathbb{1}_{\mathcal{D}})$ be an equivalence of categories. Then there are natural isomorphisms $\alpha' : \mathbb{1}_{\mathcal{C}} \xrightarrow{\simeq} GF, \beta' : FG \xrightarrow{\simeq} \mathbb{1}_{\mathcal{D}}$ satisfying the triangular identities. In particular, $F \dashv G$ and $G \dashv F$.*

Proof. Let $\alpha' = \alpha$, and define β' to be the composite

$$FG \xrightarrow{FG\beta^{-1}} FGFG \xrightarrow{F\alpha_G^{-1}} FG \xrightarrow{\beta} \mathbb{1}_{\mathcal{D}}$$

Note that

$$\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ {\scriptstyle \beta_{FG}}\downarrow & & \downarrow{\scriptstyle \beta} \\ FG & \xrightarrow[\beta]{} & \mathbb{1}_{\mathcal{D}} \end{array}$$

commutes, so $FG\beta = \beta_{FG}$. By the triangle identities we have

$$\begin{array}{ccccc} F & \xrightarrow{F\alpha} & FGF & \xrightarrow{\beta^{-1}_{FGF}} & FGFGF \\ & \searrow^{\mathbb{1}_F} & \downarrow{\scriptstyle (F\alpha)^{-1}} & & \downarrow{\scriptstyle (F\alpha_{GF})^{-1}_G=(FGF\alpha)^{-1}} \\ & & F & \xrightarrow{\beta_F^{-1}} & FGF \\ & & & \searrow^{\mathbb{1}_F} & \downarrow{\scriptstyle \beta_F} \\ & & & & F \end{array}$$

Then we also have

$$\begin{array}{ccccc} G & \xrightarrow{\alpha_G} & FGF & \xrightarrow{(GFG\beta)^{-1}} & FGFGF \\ & \searrow^{\mathbb{1}_G} & \downarrow{\scriptstyle \alpha_G^{-1}} & & \downarrow{\scriptstyle (FG\alpha_G)^{-1}=\alpha^{-1}_{GFG}} \\ & & G & \xrightarrow{(G\beta)^{-1}} & FGF \\ & & & \searrow^{\mathbb{1}_G} & \downarrow{\scriptstyle G\beta} \\ & & & & G \end{array}$$

From these diagrams, $F \dashv G$; by the same argument, α'^{-1} and β'^{-1} also satisfy the triangular identities, so $G \dashv F$. □

Lemma 3.7. *Let* $(F : \mathcal{C} \to \mathcal{D} \dashv G : \mathcal{D} \to \mathcal{C})$ *be an adjunction with counit* ε. *Then*

(i) ε *is pointwise epic iff* G *is faithful.*

(ii) ε *is an isomorphism iff* G *is full and faithful.*

Proof.

(i) Given $g : B \to B'$ in $\mathcal{D}$, the map $g\varepsilon_B : FGB \to B'$ corresponds under the adjunction to $Gg : GB \to GB'$.

Thus ε_B is epic iff G acts injectively on morphisms $B \to B'$, and so ε_B is epic for all B iff G is faithful.

(ii) By the same argument, ε_B is an isomorphism iff G acts bijectively on morphisms $B \to B'$, so ε_B is an isomorphism for all B iff G is full and faithful.

□

Definition 3.2. A **reflection** is an adjunction satisfying (ii) in the last lemma (i.e., it is an adjunction whose counit is an isomorphism). Say $\mathcal{C}' \subseteq \mathcal{C}$ is a **reflective subcategory**, or **reflective in** $\mathcal{C}$, if it's full and its inclusion map has a left adjoint, called the **reflector**.

Examples 3.2.

1. **AbGp** is reflective in **Gp**: given a group G, let G' be its commutator subgroup. Then G/G' is abelian, and any map $G \to A$ with A abelian factors through the quotient. Thus $G \to G'$ gives a left adjoint to the inclusion map, so **AbGp** is a reflective subcategory of **Gp**.

2. A group is *torsion* if all its elements have finite order. In an abelian group A, the torsion elements form a subgroup A_t; any map $B \to A$ where B is torsion factors through the inclusion, so $A \to A_t$ gives a right adjoint to the inclusion of **tAbGp** (the category of torsion abelian groups) in AbGp. We call **tAbGp** a **coreflective subcategory**, and the inclusion a **coreflection**.

3. Let **KHaus** $\subseteq$ **Top** be the category of compact Hausdorff spaces. The inclusion has a left adjoint β called the Stone-Čech compactification. We will construct it later in the course.

4. A subset $C \subseteq X$ of a topological space X is *sequentially closed* if all limit points of C are in C. Say X is *sequential* if all its sequentially closed sets are in fact closed.

   Let **Seq** be the category of sequential spaces: that is, the spaces whose sequentially-closed sets are exactly its closed sets. The inclusion map **Seq** $\to$ **Top** has a right adjoint $X \to X_s$, where X_s is X re-topologised to make all sequentially closed sets closed. The identity map $X_s \to X$ is the counit of the adjunction. Indeed, if $f : Y \to X$ is continuous and Y is sequential, then $f : Y \to X_s$ is still continuous.

5. The category **Preord** of small preorders is reflective in **Cat**. The reflector sends $\mathcal{C}$ to $\mathcal{C}/\simeq$, where $\simeq$ identifies all parallel pairs of morphisms in $\mathcal{C}$.

6. Given a topology on a set X, the poset $\mathcal{O}X$ of open subsets of X is coreflective in $\mathcal{P}X$. Indeed, the right adjoint to the inclusion is given by the interior operation $S \to S^\circ$: if U is open and A is arbitrary, then $U \subseteq A \iff U \subseteq A^\circ$.

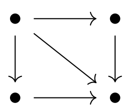
   Dually, the poset $\mathcal{C}X$ of closed subsets of X is reflective in X.

4 Limits

Definition 4.1. Let J and $\mathcal{C}$ be categories. A **diagram of shape J in $\mathcal{C}$** is a functor $D : J \to \mathcal{C}$. The objects $D(j)$ are called **vertices** of D, and morphisms $D(\alpha)$ are called **edges** of D.

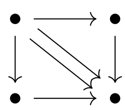
The category J is almost always small, and often finite.

Example 4.1. Suppose J is a category with 4 objects and 5 non-identity morphisms as below:



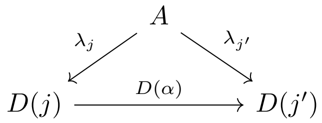
Then a diagram of shape J is a commutative square.

Suppose J' is the category with non-identity morphisms as below:



A diagram of shape J' is a square that is not necessarily commutative.

Definition 4.2. Suppose D is a diagram of shape J in a category $\mathcal{C}$. A **cone over** D comprises an object A of $\mathcal{C}$ (called the **vertex**), together with morphisms $\lambda_j : A \to D(j)$ (called the **legs**), such that, for each $\alpha : j \to j'$ in J, the triangle



commutes.

Dually, a **cocone over** D, or **cone under** D, has the same definition, except that this time $\lambda_j : D(j) \to A$.

Write **Cone**(D) for the category of cones over D. A **limit** for D is a terminal object of **Cone**(D), and a **colimit** is an initial object of **Cocone**(D), the category of cones under D.

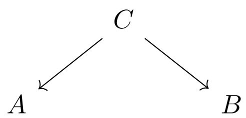
Let $\Delta : \mathcal{C} \to [J, \mathcal{C}]$ be the functor mapping A to the 'constant' diagram with all vertices A and all morphisms $\mathbb{1}_A$. Then **Cone**(D) is exactly the category $(\Delta \downarrow D)$. Hence, in particular, $\mathcal{C}$ has limits for all diagrams of shape J iff Δ has a right adjoint.

Examples 4.1.

1. Let $J = \emptyset$. A cone over the unique (empty) diagram $\emptyset \to \mathcal{C}$ is just an object of $\mathcal{C}$ (the apex), and a limit for the diagram is just a terminal object of $\mathcal{C}$.
2. Let J be the discrete two-point category.

   $$\bullet \qquad \bullet$$

   A diagram of shape J is a pair (A, B) of objects, a cone over it is a **span**

   and a limit for this cone is a (binary) product $A \times B$. Dually, a colimit for the cone under the diagram is a coproduct.
3. More generally, let DJ be a discrete category. A diagram of shape DJ is a family of objects $(A_j \mid j \in J)$, and a limit for such a diagram is a **product** $\prod_{j \in J} A_j$. Dually, a colimit for such a diagram is a **coproduct** $\sum_{j \in J} A_j$.
4. Let J be

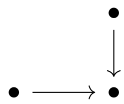
   $$\bullet \rightrightarrows \bullet$$

   A diagram of shape J is a parallel pair

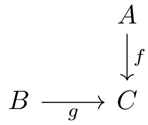
   $$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and a cone over the diagram is equivalent to a morphism $C \xrightarrow{h} A$ satisfying $fh = gh$ (since the $C \to B$ leg of the cone is redundant). A limit for such a diagram is then an equaliser of f and g.

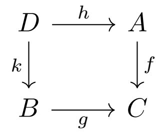
5. Let J be



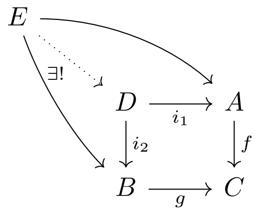
A diagram of shape J is a cospan



and a cone over this diagram is (equivalent to) a commutative square: morphisms $D \to A$ and $D \to B$ such that the respective composites with f and g commute.



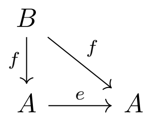
A limit for this diagram is called a **pullback** of (f, g).



The dual notion, which is a colimit of a diagram of shape J^{op}, is called a **pushout**.

Observe that we can construct pullbacks from binary products and equalisers: the pullback is the equaliser of $f\pi_1, g\pi_2 : A \times B \rightrightarrows C$.

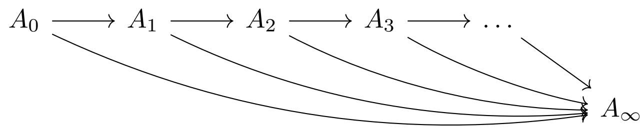
6. Let J be the monoid $\{1, e\}$ with $e^2 = e$. A diagram of shape J is a pair (A, e), where $e : A \to A$ is idempotent. A cone over this diagram is a morphism $f : B \to A$ satisfying $ef = f$, and a limit (or colimit) for this diagram is the monic (or epic) part of a splitting of e.



7. Let $J = \mathbb{N}$ be the ordered set of natural numbers. A diagram of shape J is a **direct sequence**

$$A_0 \to A_1 \to A_2 \to A_3 \to \dots$$

and a colimit for it is a universal cone of form



The colimit A_∞ is called a **direct limit**; we can think of the A_i as being successive approximations to it. Dually, a limit for a diagram of shape $\mathbb{N}^{\text{op}}$ is called an **inverse limit**.

The confusing naming convention is due to the use of such objects in topology and analytic number theory prior to the development of category theory. Indeed, infinite-dimensional CW-complexes are direct limits, and the p-adic numbers are inverse limits.

In fact, we can build all limits out of the examples we have seen so far.

Proposition 4.1.

(i) If $\mathcal{C}$ has all small products and equalizers, then $\mathcal{C}$ has all small limits.

(ii) If $\mathcal{C}$ has all finite products and equalisers, then $\mathcal{C}$ has all finite limits.

(iii) If $\mathcal{C}$ has all pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits.

Proof.

(i) Fix a small diagram $D : J \to \mathcal{C}$ (that is, J is small). We can form the products

$$P = \prod_{j \in \text{ob}\, J} D(j) \text{ and } Q = \prod_{\alpha \in \text{mor}\, J} D(\operatorname{cod} \alpha).$$

Define $f, g : P \rightrightarrows Q$ by

$$\pi_\alpha f = \pi_{\operatorname{cod} \alpha} \text{ and } \pi_\alpha g = D(\alpha)\pi_{\operatorname{dom} \alpha},$$

and let $e : L \to P$ be an equaliser of (f, g). Set $\lambda_j \coloneqq \pi_j e : L \to D(j)$.

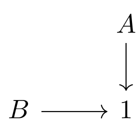
Then the λ_j form a cone: indeed, if $\alpha : j \to j'$ then

$$D(\alpha)\lambda_j = D(\alpha)\pi_j e = \pi_a lphage = \pi_\alpha f e = \pi_{j'} e = \lambda_{j'}.$$

Then a cone $(M, (\mu_j \mid j \in \text{ob}\, J))$ induces a morphism $\mu : M \to P$ satisfying $f\mu = g\mu$, and then also induces a unique $\nu : M \to L$ with $\lambda_J \nu = \mu_j$ for all j.

(ii) Same proof as (i) (replace "small" with "finite").

(iii) Let $1 \in \text{ob}\, \mathcal{C}$ be terminal. For any two objects A and B, a pullback of



is a product $A \times B$; we can then construct finite products inductively as

$$\prod_{i=1}^{n} A_n = \left(\prod_{i=1}^{n-1} A_i \right) \times A_n.$$

To form an equaliser of $f, g : A \rightrightarrows B$, consider the pullback of

$$\begin{array}{ccc} & & A \\ & & \big\downarrow {\scriptstyle (\mathbb{1}_A, f)} \\ A & \xrightarrow[(\mathbb{1}_A, g)]{} & A \times B \end{array}$$

a cone

$$\begin{array}{ccc} C & \xrightarrow[h]{} & A \\ \big\downarrow {\scriptstyle k} & & \\ A & & \end{array}$$

over this diagram satisfies $h = k$ and $fh = gk = gh$, so a limit is an equaliser for (f, g).

□

Definition 4.3. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor.

(i) G **preserves limits** of shape J if, given a diagram $D : J \to \mathcal{D}$ and a limit $(L, (\lambda_j))$ for D, the cone $(GL, (G\lambda_j))$ is a limit for GD.

(ii) G **reflects limits** of shape J if, given $D : J \to \mathcal{D}$ and a cone $(L, (\lambda_j))$ such that the cone $(GL, (G\lambda_j))$ is a limit for GD, the original cone $(L, (\lambda_j))$ is in fact a limit for D.

(iii) G **creates limits** of shape J if, for a diagram $D : J \to \mathcal{D}$ and a limit $(M, (\mu_j))$ for GD in $\mathcal{C}$, there exists a limit $(L, (\lambda_j))$ of D in $\mathcal{D}$ whose image is isomorphic to $(M, (\mu_j))$.

Note that, if we do not assume that limits of shape J actually exist, we can get vacuous definitions. If limits of shape J *do* exist in $\mathcal{C}$ and $\mathcal{D}$, then (iii) holds iff both (i) and (ii) do.

Corollary 4.2. *We can replace instances of '$\mathcal{C}$ has' in the last proposition by '$\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves', or by '$\mathcal{C}$ has and $G : \mathcal{D} \to \mathcal{C}$ creates'.*

Say $\mathcal{C}$ is **(co)complete** if it has all small (co)limits. A functor is **(co)continuous** if it preserves all small (co)limits.

Examples 4.2. 1. **Set**, **Gp**, and **Top** are complete and cocomplete since we can construct (small) products and equalisers.

2. The forgetful functor U : **Grp** $\to$ **Set** creates all small limits. Indeed, given a family of groups $(G_i \mid i \in I)$, there is a unique group structure on $\prod_{i \in I} UG_i$ for which the natural projections are homomorphisms. This group is then a product of the G_i. Similarly, an equaliser of two underlying maps is indeed a subgroup of their domain.

However, U doesn't preserve *or* reflect colimits: the coproduct of groups G and H is their free product $G * H$, which has a completely different underlying set to the coproduct of UG and UH is $UG \sqcup UH$.

3. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ preserves all small limits and colimits, but doesn't reflect either. Indeed, fix a diagram $D : J \to \mathbf{Top}$ and a limit $(L, (\lambda_j))$ for UD in $\mathbf{Set}$. Unless L is discrete, there are strictly finer topologies on L in which the λ_j are still continuous, but these are not limits of the diagram. Similarly, unless a colimit L is indiscrete, there are strictly coarser topologies such that the legs remain continuous.

4. The inclusion $\mathbf{AbGp} \to \mathbf{Gp}$ reflects binary coproducts, but doesn't preserve them. Indeed, a free product $A * B$ is non-Abelian unless either A or B is trivial. If A is trivial, then $A * B \cong B \cong A \oplus B$, so the colimits coincide.

5. The functor category $[\mathcal{C}, \mathcal{D}]$ has has all limits and colimits which exist in $\mathcal{D}$, and the forgetful functor $[\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\mathrm{ob}\, \mathcal{C}}$ creates them.

   Indeed, suppose $\mathcal{D}$ has limits of shape J, and take a diagram D of shape J in $[\mathcal{C}, \mathcal{D}]$. We can think of this as a functor $D : J \times \mathcal{D} \to \mathcal{C}$. For each $A \in \mathrm{ob}\, \mathcal{C}$, the functor $D(-, A)$ is a diagram of shape J in $\mathcal{D}$. Let $(LA, (\lambda_{j,A} : LA \to D(j, A) \mid j \in \mathrm{ob}\, J))$ be a limit for this diagram. Given $f : A \to B$ in $\mathcal{C}$, the morphisms

   $$LA \xrightarrow{\lambda_{j,A}} D(, jA) \xrightarrow{D(j,F)} D(j, B)$$

   form a cone over $D(-, B)$, so they induce a unique morphism $Lf : LA \to LB$. Functoriality of $f \to Lf$ follows from uniqueness: this is the unique way of making the mapping $A \to LA$ into a functor $\mathcal{C} \to \mathcal{D}$ which makes the $\lambda_{j,-}$ into natural transformations.

   Then $(L, (\lambda_{j,-}))$ is indeed a limit for D: given any cone $(M, (\mu_{j,-} \mid j \in \mathrm{ob}\, J))$ over D in $[\mathcal{C}, \mathcal{D}]$, we get a unique map $\nu_A : MA \to LA$ for each A satisfying $\lambda_{j,A} \nu_A = \mu_{j,A}$ for all j. This map is natural in A by uniqueness of factorisation through limits.

In any category, a morphism $fA \to B$ is monic iff the square below is a pullback.

$$\begin{array}{ccc} A & \xrightarrow{\mathbb{1}_A} & A \\ {\scriptstyle \mathbb{1}_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

Therefore, if a category $\mathcal{D}$ has pullbacks, then any monomorphism in $[\mathcal{C}, \mathcal{D}]$ must be a pointwise monomorphism, since the pullback above must be constructed pointwise by the last example.

In particular, $\mathbf{Set}$ has pushouts, so the epimorphisms in $[\mathcal{C}, \mathbf{Set}]$ are exactly the pointwise epimorphisms.

Lemma 4.3. *Suppose $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint. Then G preserves any limits which exist in $\mathcal{D}$.*

We give two proofs. The first is more conceptual, but needs additional hypotheses on $\mathcal{C}$ and $\mathcal{D}$. The second is more technical, but more general.

Proof 1. Suppose $(F \dashv G)$, and suppose further that both $\mathcal{C}$ and $\mathcal{D}$ both have limits of shape J. Then the square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ [J,\mathcal{C}] & \xrightarrow[{[J,F]}]{} & [J,D] \end{array}$$

commutes, where $[J,F]$ is F applied to diagrams of shape J.

But $([J,F] \dashv [J,G])$, and all functors in this diagram have right adjoints, so

$$\begin{array}{ccc} [J,\mathcal{D}] & \xrightarrow[{[J,G]}]{} & [J,\mathcal{C}] \\ \downarrow{\scriptstyle\lim_J} & & \downarrow{\scriptstyle\lim_J} \\ \mathcal{D} & \xrightarrow[G]{} & \mathcal{C} \end{array}$$

commutes (up to isomorphism). But this says exactly that G preserves limits of shape J. □

Proof 2. □

This has a converse: as we will see we can find left adjoints to functors preserving limits.

Lemma 4.4. *Suppose G has limits of shape J, and $G : \mathcal{D} \to \mathcal{C}$ preserves them. Then, for each $A \in \operatorname{ob}\mathcal{C}$, $(A \downarrow G)$ has limits of shape J, and the forgetful functor $U : (A \downarrow G) \to \mathcal{D}$ creates them.*

Proof. Fix a diagram $D : J \to (A \downarrow G)$. Write $D(j) = (UD(j), f_j)$, where $f_j : A \to GUD(j)$. Since the edges of D are morphisms in $(A \downarrow G)$, the f_j form a cone over GUD.

Let $(L, (\lambda_j : L \to UD(j)))$ be a limit for UD; then $(GL, (G\lambda_j))$ is a limit for GUD, so there is a unique $g : A \to GL$ such that $(G\lambda_j)g = f_j$ for all j. Therefore (L, g) is the unique lift of L to an object of $(A \downarrow G)$ making the λ_j into morphisms.

This lift is a limit by the same argument as for $[\mathcal{C}, \mathcal{D}]$ (see the last example above). □

Lemma 4.5. *An initial object of a category $\mathcal{C}$ is precisely a limit of the diagram $\mathbb{1}_{\mathcal{C}}$ of shape $\mathcal{C}$.*

Proof. Suppose I is initial. The (unique) morphisms $f_A : I \to A$ for $A \in \operatorname{ob}\mathcal{C}$ form a cone over $\mathbb{1}_{\mathcal{C}}$; for any other cone with apex J and legs $g_A : J \to A$, the map g_I factors the g_A through the f_A uniquely.

Conversely, suppose $(I, (f_A \mid A \in \operatorname{ob}\mathcal{C}))$ is a limit for $\mathbb{1}_{\mathcal{C}}$. We want to show the f_A is the unique morphism $f_A : I \to A$. Indeed, for any $g : I \to A$, we have $gf_I = f_A$. In particular, $f_A f_I = f_A$ for all A, so f_I factors the limit cone through itself. By uniqueness of factorisation, $f_I = \mathbb{1}_I$, so $g = f_A$. □

Combining these two lemmas with the result relating adjoints with comma categories, we have proved the '*Primitive Adjoint Functor Theorem*': if $G : \mathcal{D} \to \mathcal{C}$ preserves *all* limits, then G has a left adjoint. This is not very practical (limits can be big).

Theorem 4.6 (General Adjoint Functor Theorem)**.** *Suppose $\mathcal{D}$ is complete and locally small. Then $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint iff it preserves all small limits and satisfies the* ***solution set condition*** *(SSC): for any $A \in \mathrm{ob}\,\mathcal{C}$, there is a set of morphisms $\{f_i : A \to GB_i \mid i \in I\}$ such that every $f : A \to GB$ factors through some f_i as*

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg} GB.$$

We call a set of morphisms as in the statement **collectively weakly initial**. 'Weakly' just means a uniqueness condition is dropped. For example, the following proof uses the concept of a **weakly initial** object I: an object with *at least one* morphism to each other object.

Proof.
$\Rightarrow$: If $(F \dashv G)$, then G preserves all limits, so $\{\eta_A : A \to GFA\}$ is a singleton solution set.

$\Leftarrow$: We have shown that each $(A \downarrow G)$ is complete, and that comma categories inherit local smallness from $\mathcal{D}$. It therefore remains to show that, if $\mathcal{A}$ is complete and locally small, and has a weakly initial set $\{S_i \mid i \in I\}$ of objects, then $\mathcal{A}$ has an initial object.

Indeed, take such a $\mathcal{A}$. Form $P = \prod_{i \in I} S_i$; then P is also weakly initial. Now, form the limit of the diagram

$$P \rightrightarrows\!\!\rightrightarrows P$$

whose edges are the endomorphisms of P. Suppose $i : I \to P$ is a morphism; then I is also weakly initial. We will show that, in fact, I is initial. Indeed, suppose we have $f, g : I \to T$; form the equaliser $e : E \to I$ of (f, g); then there is some $p : H \to E$. Now, $ieh : P \to P$, but we also have $\mathbb{1}_P : P \to P$, so $iehi = \mathbb{1}_P i = i$. But i is monic, so $ehi = \mathbb{1}_I$, so e is epic and $f = g$. □

Examples 4.3.

1. Consider the forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$; suppose we have never seen the free group construction before, and want to prove this has a left adjoint. Indeed, $\mathbf{Gp}$ is complete, U preserves all small limits, and $\mathbf{Gp}$ is locally small. It remains to verify the SSC.

   Indeed, fix A and $f : A \to UG$. This factors as $A \to UG' \hookrightarrow UG$, where G' is the subgroup of G generated by $\mathrm{im}\, f$; note G' has cardinality at most $\max\{\aleph_0, \mathrm{card}\, A\}$. Fix a set B of this cardinality, and consider all subsets $B' \subseteq B$, all possible group structures on the B', and all possible mappings $A \to B'$. This gives a solution set at A.

2. Let $\mathbf{CLat}$ be the category of complete lattices (posets with all joins and meets). The forgetful functor $U : \mathbf{CLat} \to \mathbf{Set}$ creates all small limits, so $\mathbf{CLat}$ is complete; it is also locally small. But Hales proved that, for any

cardinal κ, there is a 3-generator complete lattice of cardinality $\geq \kappa$, so U has no solution set for the 3-element set $\{a, b, c\}$. Therefore U has no left adjoint.

Definition 4.4. A **subobject** of an object A of $\mathcal{C}$ is a monomorphism $m : A' \hookrightarrow A$. The class of subobjects of A forms a preorder $\mathrm{Sub}_{\mathcal{C}}A$, with $m \leq m'$ if m factors through m'.

$\mathcal{C}$ is **well-powered** if $\mathrm{Sub}_{\mathcal{C}}A$ is equivalent to a small category for every $A \in \mathrm{ob}\,\mathcal{C}$: that is, if, for every $A \in \mathrm{ob}\,\mathcal{C}$, there is a *set* of subobjects of A meeting every isomorphism-class of subobjects. Dually, if $\mathcal{C}^{\mathrm{op}}$ is well-powered, then $\mathcal{C}$ is **well-copowered**.

Example 4.2. By the power-set axiom, **Set** is well-powered. Since subobjects in **Gp**, **Rng**, etc., are inclusions, these categories are well-powered for the same reason.

Theorem 4.7. *Take a pullback square*

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

with f monic. Then k is monic.

Proof. Take $l, m : D \rightrightarrows P$ with $kl = km$. Then $fhl = gkl = gkm = fhm$; since f is monic, $hl = hm$. Since l and m are factorisations for the same cone through P, $l = m$. □

Theorem 4.8 (Special Adjoint Functor Theorem)**.** *Suppose $\mathcal{C}$ and $\mathcal{D}$ are locally small, and $\mathcal{D}$ is complete and well-powered, with a coseparating set of objects. Then $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint iff it preserves all small limits.*

Proof.
$\Rightarrow$: Already done.

$\Leftarrow$: Let $A \in \mathrm{ob}\,\mathcal{C}$. As in the general AFT, $(A \downarrow G)$ is complete and locally small. It is also well-powered. Indeed, the subobjects of (B, f) are subobjects $m : B' \hookrightarrow B$ such that f factors through Gm. Then, since G preserves pullbacks, it preserves monomorphisms (by the last lemma). Also, if $\{B_i \mid i \in I\}$ is a separating set for $\mathcal{D}$, then the set $\{(B_i, f) \mid i \in I, f \in \mathcal{C}(A, GB_i)\}$ is coseparating in $(A \downarrow G)$.

It therefore remains to show that a category $\mathcal{A}$ which is complete, locally small, well-powered, and has a coseparating set $\{S_i \mid i \in I\}$ has an initial object.

Form $P = \prod_{i \in I} S_i$, and consider the diagram

$$\begin{array}{ccc} P'' & & P' \\ & \searrow & \downarrow \\ \cdots & \hookrightarrow & P \end{array}$$

whose edges are the members of a representative set of subobjects of P. If I is the apex of a limit cone, then, by the argument in the last lemma, the legs $I \to P'$ are all monic. In particular, $I \hookrightarrow P$ is monic, so it's a least subobject of P. If we have $f, g : I \rightrightarrows T$, their equaliser $e : E \hookrightarrow I$ is a subobject of P contained in I, so it's an isomorphism. Hence $f = g$. It remains to show that I is weakly initial.

Given any object T of $\mathcal{A}$, form the product $Q = \prod_{(i,f)} S_i$, where the product ranges over pairs (i, f) with $i \in I$ and $f : T \to S_i$. Define $g : T \to Q$ by $\pi_{(i,f)} g = f$. Since the S_i are coseparating, g is monic. We also have $h : P \to Q$ defined by $\pi_{(f,i)} h = \pi_i$. Now, form the pullback

$$\begin{array}{ccccc} I & \dashrightarrow & U & \longrightarrow & T \\ & \searrow & \big\downarrow k & & \big\downarrow g \\ & & P & \xrightarrow{h} & Q \end{array}$$

By the last lemma, k is monic, so there is a map $I \to U$, and so $I \to T$. Hence I is weakly initial, and so initial. □

Example 4.3. Consider the inclusion $I : \mathbf{KHaus} \to \mathbf{Top}$. By Tychonoff's theorem, **KHaus** is closed under products in **Top**. It's also closed under equalisers: given $f, g : X \rightrightarrows Y$ in **KHaus**, their equaliser is closed in X (since Y is Hausdorff), and hence compact. Hence **KHaus** is complete and I preserves all limits. Then **Top** and **KHaus** are both locally small, and **KHaus** is well-powered since any subobject of $X \in \operatorname{ob} \mathbf{KHaus}$ is (isomorphic to) the inclusion of a closed subspace. Finally, **KHaus** has a coseparator $[0, 1]$ by Uryson's lemma. Hence, by special AFT, I has a left adjoint.

This is a categorical guarantee of the existence of the Stone-Čech compactification. Indeed, Čech's original construction was very similar to the one in the proof of special AFT: given a space X, he formed $P = \prod_{f:X\to[0,1]} [0, 1]$ and defined the canonical map $g : X \to P$ by $\pi_f g = f$. Then he took βX to be the closure of $\operatorname{im} g$: this is exactly the smallest subobject of (P, g) in $(X \downarrow I)$.

Note that the hypotheses of the general adjoint functor theorem also hold in this case: given a space X in **Top**, any $f : X \to IY$ factors through $IY' \to IY$, where Y' is the closure of $\operatorname{im} f$. If Y' is Hausdorff and has a dense subspace of cardinality at most κ, then $\operatorname{card} Y' \leq 2^{2^\kappa}$, so we get a *set* of representatives of isomorphism classes of such objects.

5 Monads

Suppose we have an adjunction $(F \dashv G)$ between $\mathcal{C}$ and $\mathcal{D}$, but we don't know what $\mathcal{D}$ is. How much of $(F \dashv G)$ can we describe in terms of $\mathcal{C}$ alone?

We have the functor $T = GF : \mathcal{C} \to \mathcal{C}$, and the unit $\eta : 1_{\mathcal{C}} \to T$. We also have the natural transformation $\mu := G\varepsilon_F : TT = GFGF \to GF = T$.

The triangular identities

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow^{\mathbb{1}} & \downarrow{\scriptstyle \varepsilon_F} \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & \searrow^{\mathbb{1}} & \downarrow{\scriptstyle G\varepsilon} \\ & & G \end{array}$$

give us the diagrams

$$(1) \quad \begin{array}{ccc} T & \xrightarrow{T\eta} & TT \\ & \searrow^{\mathbb{1}_T} & \downarrow{\scriptstyle \mu} \\ & & T \end{array} \qquad (2) \quad \begin{array}{ccc} T & \xrightarrow{\eta_T} & TT \\ & \searrow^{\mathbb{1}_T} & \downarrow{\scriptstyle \mu} \\ & & T \end{array}$$

Also, naturality of ε yields the square

$$(3) \quad \begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ \downarrow{\scriptstyle \mu_T} & & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow{\mu} & T \end{array}$$

We abstract this:

Definition 5.1. A **monad** on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$ equipped with a **unit** $\eta : \mathbb{1}_G \to T$ and a **multiplication** $\mu : TT \to T$ satisfying the diagrams (1), (2) and (3) above. We write $\mathbb{T} = (T, \eta, \mu)$.

By construction, every adjunction gives rise to a monad. We can think of the commutative diagrams a monad satisfies as encoding the unit and associativity laws of a monoid.

Example 5.1. Let M be a monoid in **Set**. The functor $M \times (-) : \mathbf{Set} \to \mathbf{Set}$ has a monad structure. Indeed, define $\eta_A : A \to M \times A$ by $\eta_A(a) = (1, a)$ and $\mu_A : M \times M \times A \to M \times A$ by $\mu_A(m, m', a) = (mm', a)$. These are obviously natural in A (since they don't act on A at all), and the commutative diagrams follow from the unit and associative laws in M.

Take a monad $\mathbb{T}$ on $\mathcal{C}$. Is there an adjunction inducing $\mathbb{T}$? The answer is yes – we have 2 explicit constructions. We will now study them both.

Definition 5.2. Let $\mathbb{T}$ be a monad on $\mathcal{C}$. A **Eilenberg-Moore algebra** for $\mathbb{T}$, or **$\mathbb{T}$-algebra**, is a pair (A, α), where $A \in \mathrm{ob}\,\mathcal{C}$ and $\alpha : TA \to A$ satisfies the diagrams (4) and (5) below.

$$(4) \quad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow^{\mathbb{1}_A} & \downarrow{\scriptstyle \alpha} \\ & & A \end{array} \qquad (5) \quad \begin{array}{ccc} TTA & \xrightarrow{T\alpha} & TA \\ \downarrow{\scriptstyle \mu_\alpha} & & \downarrow{\scriptstyle \alpha} \\ TA & \xrightarrow{\alpha} & A \end{array}$$

A **$\mathbb{T}$-algebra homomorphism** $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ satisfying diagram (6) below.

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{f} & B \end{array}$$

Write $\mathcal{C}^{\mathbb{T}}$ for the category of $\mathbb{T}$-algebras and homomorphisms.

Proposition 5.1. *The forgetful functor $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ has a left adjoint $F^{\mathbb{T}}$, and the adjunction induces $\mathbb{T}$.*

Given an adjunction $(F \dashv G) : \mathcal{C} \leftrightarrows \mathcal{D}$, we can replace $\mathcal{D}$ by the full 'image' subcategory $\mathcal{D}' \subseteq \mathcal{D}$ whose objects are those of form FA; we can then make distinct isomorphic copies of those objects in $\mathcal{D}'$ on which F is not injective, and copy the morphisms in the obvious way. Changing G to map the copies in the natural way, these changes do not affect $GF = T$.

Therefore, in constructing an adjunction from T, we can assume wlog that F is bijective on objects. But then morphisms $FA \to FB$ in $\mathcal{D}$ must correspond to morphisms $A \to TB$ in $\mathcal{C}$.

Definition 5.3. Let $\mathbb{T}$ be a monad on $\mathcal{C}$. The **Kleisli category** $\mathcal{C}_{\mathbb{T}}$ has objects $\operatorname{ob}\mathcal{C}_{\mathbb{T}} = \operatorname{ob}\mathcal{C}$; morphisms $f : A \rightsquigarrow B$ in $\mathcal{C}_{\mathbb{T}}$ are morphisms $f : A \to TB$ in $\mathcal{C}$. We write morphisms in $\mathcal{C}_{\mathbb{T}}$ using $\rightsquigarrow$. The identity $A \rightsquigarrow A$ is $\eta_A : A \to TA$, and the composite $gf : A \rightsquigarrow B \rightsquigarrow C$ is

$$A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC.$$

The η_A act as the identity since we have

And, given $hgf : A \rightsquigarrow B \rightsquigarrow C \rightsquigarrow D$, we have

so composition is associative. Hence $\mathcal{C}_{\mathbb{T}}$ really is a category.

Proposition 5.2. *There is an adjunction $(F_{\mathbb{T}} : \mathcal{C} \to \mathcal{C}_{\mathbb{T}} \dashv G_{\mathbb{T}} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C})$ inducing $\mathbb{T}$.*

Proof. Let $F_{\mathbb{T}}A = A$, and $F_{\mathbb{T}}(f : A \to B) = \eta_B f : A \to TB$. This preserves identity by definition, and composition since the diagram below commutes.

Let $G_{\mathbb{T}}A = TA$, and let $G_{\mathbb{T}}(f : A \rightsquigarrow B) = \mu_B(Tf) : TA \to TTB \to TB$. This preserves identity by (1); for the same reason, $G_{\mathbb{T}}F_{\mathbb{T}} = T$. Then $\mathbb{G}_{\mathbb{T}}$ preserves composition by the diagram below.

It remains to show $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$. Indeed, take η to be the unit of the adjunction, and let the counit $\varepsilon_A : TA \rightsquigarrow A$ be $\eta_A = \mathbb{1}_{T_A}$.

Then $F_{\mathbb{T}}G_{\mathbb{T}}$ is

$$TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{\eta_{TB}} TB,$$

so its composite with μ_B is $\mu_B(Tf)$. This gives the diagram.

The triangular identities for η and ε hold by (2), so $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$; then also $G_{\mathbb{T}}\varepsilon_{F_{\mathbb{T}}A} = \mu_A$, so the adjunction induces $\mathbb{T}$. □

Given a monad $\mathbb{T}$ on $\mathcal{C}$, write $\mathrm{Adj}_{\mathbb{T}}$ for the category whose objects are adjunctions $F, G : \mathcal{C} \leftrightarrows \mathcal{D}$ inducing $\mathbb{T}$, and whose morphisms $(F, G : \mathcal{C} \leftrightarrows \mathcal{D}) \to (F', G' : \mathcal{C} \leftrightarrows \mathcal{D})$ are functors $K : \mathcal{D} \to \mathcal{D}'$ such that $KF = F'$ and $G'K = G$.

Theorem 5.3. *The Eilenberg-Moore adjunction $(F^{\mathbb{T}} \dashv G^{\mathbb{T}})$ is an initial object of $\mathrm{Adj}_{\mathbb{T}}$, and the Kleisli adjunction $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ is a terminal one.*